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**OPTIMIZATION, SYSTEM ANALYSIS, AND OPERATIONS RESEARCH**

# On the Emergence of Binary Mapping Systems

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**Abstract**—The emergence effect is considered in terms of mappings realized by abstract systems that describe observable physical manifestations. The emergence phenomenon is interpreted in terms of the expansion of the class of realized mappings when elements are combined into a system or new connections between elements arise. The paper considers the problem of describing emergent properties in models of systems that realize binary mappings. The structure of interaction of elements in such systems is described by a finite oriented graph. Classes of mappings are studied for set-theoretical operations on such systems. A "superadditive" expansion of the class of mappings is shown when systems are combined. The emergence coefficient is introduced and substantiated. Lower and upper bounds for this coefficient are proved.

*Keywords*: binary mapping, system, set-theoretic operations, emergence, emergence coefficient

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## 1. INTRODUCTION

In systems analysis, the concept of emergence is one of the most interesting properties of complex systems. In the standard formulation, emergence is the appearance of new properties and qualities that are not inherent in the elements that make up the system. The term "system effect" is also used as a synonym for emergence. The history of the origin and development of the concept of emergence within the framework of general systems theory, as well as the basic ideas underlying this phenomenon, are presented in [1]. In particular, with reference to [2], it is indicated that emergent behavior can be understood based on the nature and behavior of its parts plus knowledge of how these parts interact.

A large number of works are devoted to the philosophical and general methodological understanding of this phenomenon as a whole [3]. In [4], the categories of strong and weak emergence are substantiated; these concepts are quite conditional, and it is difficult to draw a clear boundary between them. Moreover, it is believed that emergence can manifest itself on a continuum from "weak" to "strong" [5]. Emergence is considered strong if there is no acceptable theory that could explain or derive the behavior of the system based on the properties or behavior of its components. On the contrary, emergence is weak if the dependence between the behavior of the components and the behavior of the system is clearly observed. In [6], an attempt is made to mathematically prove the possibility of defining the concept of strong emergence. The paper [7] is devoted to the formalization of the concept of emergence based on entropy. It is argued that there is a connection between entropy and emergence in complex systems in the sense that an increase in entropy is an indicator of the possible occurrence of emergent events and states. In conclusion, the authors formulate the question of how to express the entropy of a system through the entropy of its constituent subsystems.

Attempts to give a formal description of such effects in specific physical or engineering systems cause significant difficulties, so there are far fewer examples of such studies. Here it is worth noting the work [8], which examines the occurrence of emergent effects in artificial systems, as well as the work [9], devoted to the study of principles and approaches to the construction of engineering systems with such properties. Specific examples of the manifestation of the system effect in real physical systems can be found, for example, in [10–13].

Due to the lack of a clear understanding, emergence is defined differently in different studies. For example, in [14, 15] it is understood as the ability of interacting autonomous agents to form coordinated behavior without external control, but only on the basis of internal interaction. A similar view is reflected in [16], which examines the processes of selforganization and emergence in the formation of a common vocabulary at the primary stages of the formation of human communities. The main idea is also based on modeling the interaction of agents acting on the basis of a small number of simple rules, which leads to emergent effects.

The study of emergence is closely related to the study of processes that lead to complex behavior of a system with relatively simple behavior of its elements. Perhaps the formal justification of such processes even in relatively narrow specific cases of such systems is the key issue in understanding the nature of emergence as such. These issues are central to the publication [17].

The system effect is found in systems of various natures – polymetallic compounds, chemical processes, collective decision-making in social structures, etc. The diversity of its manifestations predetermines the formal methods that are most suitable for research in each case. These may be equations describing the elastic interaction of solids, laws of chemical reactions, or algorithms for making compromise decisions, in particular, algorithms for group control in multiagent systems. The study of system effects involves the preliminary construction of a formal model that should correctly predict the behavior of the object or process under permissible impacts on it.

It is generally accepted that for the existence of any system, system-forming factors are necessary, which contribute to the formation of the system, are external in relation to its elements, and are not caused by the need for unification [18]. In many cases, the role of such factors is played by connections between elements, the nature of which is determined by the nature of the elements that form the system.

The basic assumption that motivated this work is that a fairly universal way to study emer-gence is to study the mappings realized by a complex system depending on the mappings realized by the elements of the system and on the state of the system-forming factors, i.e., the structure of the connections between the elements of the system. This is almost obvious in the case of information processing systems, including those that include people.

Below we consider a class of systems implementing binary mappings. In a certain sense, they can be considered as automaton mappings. Despite their apparent simplicity, automaton mappings are in many cases not inferior to classical algebraic structures in terms of their breadth of application and expressiveness. It is also known that semigroups, groups and rings of automata, as well as their functional systems, have contributed to the solution of a number of difficult abstract mathematical problems.

Models of binary mapping systems do not provide a complete explanation of all processes of system emergence. However, they demonstrate important properties that are inherent in this phenomenon. For example, the so-called "superadditive" effect that occurs when combining systems, as well as the emergence of limit cycles in a set of states. In physical systems, limit cycles correspond to attractors that can be considered as possible macrostates in multiscale systems. In the case of binary mapping systems, it becomes possible to move to strictly formal models and obtain a number of quantitative characteristics of processes that reflect emergence. It should be noted that superadditive effects on classes of binary mappings, according to the almost generally accepted terminology and classification, can be attributed to the class of weak emergence [4–6], since there is a high level of traceability between the functions of the elements and the functions of the system as a whole.

The paper is structured as follows. Section 2 provides the basic definitions, notations, and terms that are used further in the text. Sections 3 and 4 contain the derivation of the basic relations describing the realized binary mappings in systems with the structure of a complete graph and an arbitrary directed graph, respectively. Section 5 proves statements describing the classes of binary mappings that are obtained as a result of set-theoretic operations on systems. The resulting system effect is also described. Section 6 is devoted to the definition and justification of the emergence coefficient for binary systems, as well as the proof of two-sided estimates for it.

## 2. BASIC DEFINITIONS AND NOTATIONS

We will use the *Kronecker product* $A \otimes B$ of matrices $A$ and $B$, namely, if $A = (a_{ij})$ is a matrix of dimensions $m \times n$, and $B$ is a matrix of dimensions $p \times q$, then by definition the matrix $A \otimes B = (a_{ij}B)$ and obviously has dimensions $mp \times nq$. The definition and properties of the Kronecker product can be found, for example, in [19]. In particular, we will need the equality $(A \otimes B)(C \otimes D) = AC \otimes BD$ connecting the ordinary and Kronecker products, where matrices $A$, $B$, $C$ and $D$ have sizes for which all products are defined. This property can be generalized to an arbitrary number of factors. For convenience, we will denote the Kronecker product of matrices $A_1, A_2, \dots, A_k$ by $A_1 \otimes A_2 \otimes \dots \otimes A_k = \overset{k}{\underset{i=1}{\otimes}} A_i$. In particular, we will write the Kronecker $k$th power of a matrix $A$ as $\underbrace{A \otimes \dots \otimes A}_{k} = A^{[k]}$. It is important to take into account the non-commutativity of the Kronecker product.

The *Kronecker extension of a vector* $a = (a_1\ a_2 \dots a_n)^\top$ is the vector

$$\bar{a} = (1\ \ a_1\ \ a_2\ \ (a_1a_2)\ \dots\ (a_1a_2\dots a_n))^\top = \begin{pmatrix} 1 \\ a_n \end{pmatrix} \otimes \begin{pmatrix} 1 \\ a_{n-1} \end{pmatrix} \otimes \dots \otimes \begin{pmatrix} 1 \\ a_1 \end{pmatrix}.$$

The *Kronecker extension of a matrix A by rows* is the matrix $\bar{A}$, each row of which, considered as a vector, is the Kronecker extension of the corresponding row of the matrix $A$. The *Hadamard product* of matrices $A$ and $B$ is the matrix $A * B$, obtained as a result of element-wise multiplication of these matrices [19]. This product is defined for matrices of the same size and is a commutative operation. The matrix is called *monomial by columns* if each of its columns consists of zeros except for one element, which is equal to one.

Below we will need some basic definitions from graph theory. Let $H = (V, E)$ be a directed graph, where $V = \{v_1, v_2, \dots, v_n\}$ is the set of vertices and $E = \{e_1, e_2, \dots, e_p\} \subseteq V \times V$ is the set of arcs. The graph is allowed to contain loops, i.e., arcs of the form $e = (v, v)$. If $v_i \in V$, then $\rho_i$ will denote the *indegree* of the vertex $v_i$, i.e., the number of arcs of the form $(v, v_i)$ entering this vertex, so that $\sum_{i=1}^n \rho_i = |E|$. A graph with $n$ vertices that contains all possible arcs and loops, i.e., when $E = V \times V$, $|E| = n^2$, is called a *complete directed graph with loops* (or simply a *complete graph* for short), and will be denoted by $K_n$. If $E = \varnothing$, then such a graph is called a *null graph* and is denoted by $O_n$.

The *adjacency matrix* of a directed graph $H$ is a square matrix $R = (r_{ij})$ of size $n \times n$, in which $r_{ij} = 1$, if $(v_i, v_j) \in E$, and $r_{ij} = 0$ otherwise. The *union* and *intersection* of graphs $H_1 = (V, E_1)$ and $H_2 = (V, E_2)$ with the same set of vertices are graphs $H_1 \cup H_2 = (V, E_1 \cup E_2)$ and $H_1 \cap H_2 = (V, E_1 \cap E_2)$, respectively.

## 3. SYSTEMS WITH COMPLETE GRAPH STRUCTURE

By a *system* we mean a set of $n$ interacting functional elements that operate in discrete time $t = 0, 1, 2, \ldots$ and can be in one of two *states* – 0 or 1. The state $z_i$ of the $i$th element at a given time $t+1$ is determined by a Boolean function $z_i(t+1) = f_i\,(x_1(t+1), \ldots, x_m(t+1); z_1(t), \ldots, z_n(t))$ that depends on two groups of arguments. The arguments of the first group are external arguments $x_1(t+1), \ldots, x_m(t+1)$ that are binary external signals, the arguments of the second group are internal arguments $z_1(t), \ldots, z_n(t)$ that are the states of the corresponding elements of the system at a given time $t$. We will call the functions $z_i$, $i = 1, \ldots, n$, the *state functions* of the elements, and the ordered set of states of all elements — the *state of the system.* We assume that the system does not depend on random factors.

The sequence of states of the system can be considered as a binary mapping of the sequence of external arguments and previous states. In general, the functions implemented by elements may not depend on the states of some other elements, which is determined by the structure of connections between elements. Therefore, the sequence of states depends on: 1) the functions implemented by each element of the system, and 2) the structure of connections between elements. If the system contains $n$ elements with $m$ inputs, then each element can be described by one of the $2^{m+n}$ Boolean functions.

The structure of the system is conveniently defined using a directed graph, the vertices of which are in one-to-one correspondence with the elements of the system, and the arcs correspond to the connections between the outputs of some elements and the inputs of others. Since the state of each element can also depend on its own state, the system graph can have loops. The described system can be considered as a synchronous *sequential* logical network implementing an *automaton* mapping [20].

Let us consider a system of $n$ elements with structure $K_n$. The mapping realized by it can be described by a system of state functions, in which the right-hand sides are some Boolean functions:

$$z_i(t+1) = f_i\,(x_1(t+1), \ldots, x_m(t+1), z_1(t), \ldots, z_n(t)), \quad i = 1, \ldots, n.$$

Next, we will consider the case when there are no external arguments, i.e., $z_i(t+1) = f_i\,(z_1(t), \ldots, z_n(t))$, $i = 1, \ldots, n$. By analogy with differential equations, such a system can be called autonomous, describing its "own free movement" or, in the terminology of systems analysis, a closed system demonstrating its behavior depending on the initial state $z_1(0), \ldots, z_n(0)$. We will describe the mappings realized by such a system for all possible state functions of the system's elements.

As is known [21–23], any Boolean function of $n$ variables can be uniquely defined by its *arithmetic representation*:

$$\begin{aligned} z_i(t+1) = f_i(z_1(t), \ldots, z_n(t)) = a_0^i + a_1^i z_1(t) + a_2^i z_2(t) + a_{12}^i z_1(t) z_2(t) + \ \ldots \\ + a_{k_1\ldots k_p}^i z_{k_1}(t) \cdot \ldots \cdot z_{k_p}(t) + \ldots + a_{1\ldots n}^i z_1(t) \cdot \ldots \cdot z_n(t), \qquad i = 1, \ldots, n, \end{aligned} \tag{1}$$

where $a_0^i, a_1^i, a_2^i, a_{12}^i, \ldots, a_{k_1\ldots k_p}^i, \ldots, a_{1\ldots n}^i$ are integers that uniquely define the function $f_i$.

Let us consider the Kronecker extension of the vector $z = (z_1\ z_2\ \ldots\ z_n)^\top$:

$$\bar{z} = (1\ \ z_1\ \ z_2\ \ (z_1 z_2)\ \ \ldots\ \ (z_1 z_2 \ldots z_n))^\top = \begin{pmatrix} 1 \\ z_n \end{pmatrix} \otimes \begin{pmatrix} 1 \\ z_{n-1} \end{pmatrix} \otimes \ \ldots \ \otimes \begin{pmatrix} 1 \\ z_1 \end{pmatrix}.$$

Then the system of relations (1) can be written in matrix form:

$$z(t+1) = A\bar{z}(t), \tag{2}$$

here $A$ is the matrix of the size $n \times 2^n$ of the corresponding coefficients.

Equality (2) can be written differently. The vector $\bar{z}(t)$ takes $2^n$ different values, and each of them is assigned an image under the mapping $A$. Therefore, we can write $2^n$ equalities (2) corresponding to different $\bar{z}(t)$. It is easy to prove by induction that all possible $2^n$ values of the vector $\bar{z}(t)$, written in the natural order, form a matrix $Q_n = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{[n]}$. Let us denote by $B$ the matrix whose columns are the images of the corresponding vectors $\bar{z}(t)$. Obviously, this matrix completely and uniquely describes the binary mapping realized by the system. Then all possible $2^n$ equalities (2) can be written compactly as $B = AQ_n$. Since $Q_n^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}^{[n]}$, we immediately obtain

$$A = BQ_n^{-1}. \tag{3}$$

This formula gives an explicit expression for the coefficients of the arithmetic representation of the state functions of all elements of the system that realizes the binary mapping specified by the matrix $B$. The uniqueness of solution (3) means that any mapping can be realized in a system with structure $K_n$. In other words, the *class of mappings* realized in such a system coincides with the set of all possible mappings. If the structure of the system differs from $K_n$, then this will no longer be true.

Next, we describe the classes of realizable binary mappings depending on the structure of the system, defined by an arbitrary directed graph (not necessarily a graph $K_n$).

## 4. SYSTEMS WITH ARBITRARY STRUCTURE AND CLASSES OF MAPPINGS

Let us consider the general case when the system has an arbitrary structure defined by a directed graph $H = (V, E)$. The state function of the $i$th element depends on the state of the $j$th element if two conditions are simultaneously met: 1) there is a connection between the input of the $i$th element and output of the $j$th element and 2) the value of the state of the $j$th element is an essential variable for the $i$th element. The first condition is determined by the structure of the system, and the second by the state function. This reflects the relationship between the intra-element and supra-element organization of the system. A system with a structure $K_n$ is maximally "informed," since each element has information about the states of all other elements (including itself) and can be configured to perform any function.

Let us characterize the class of mappings realized by a system with an arbitrary structure $H$. Let $R = (r_{ij})$ be the adjacency matrix of the graph $H$. Using the arithmetic representation (1), we obtain a matrix relation $z(t + 1) = A\bar{z}(t)$ similar to (2), where, as before, $A$ is a matrix, each row of which is formed by the coefficients of the corresponding arithmetic representations.

However, in the general case $z_i(t + 1)$ does not depend on all values of $z_j(t)$, $j = 1, \ldots, n$, but only on some of them, determined by the structure of the system. More precisely, $z_i(t + 1)$ essentially depends on $z_j(t)$ if $r_{ji} = 1$, and, accordingly, does not depend if $r_{ji} = 0$. Thus, all information about the structural dependence of the $i$th element on the other elements is contained in the $i$th column of the matrix $R$. If $z_i(t + 1)$ does not structurally depend on $z_j(t)$, then we can assume that the coefficient $a_j^i$ in the expansion is equal to zero or, in the general case, we can write $a_j^i = r_{ji}u_j^i$. It is clear that in the remaining terms of the expansion, for example, $a^i_{k_1\ldots k_2} z_{k_1}(t) \cdot \ldots \cdot z_{k_2}(t)$, the coefficient $a^i_{k_1\ldots k_2}$ is also equal to zero if at least one of the connections of the $k_1$th, …, $k_p$th elements with the input of the $i$th element is absent. This is equivalent to the equality: $a^i_{k_1\ldots k_p} = r_{k_1 i} \cdot \ldots \cdot r_{k_p i} u^i_{k_1\ldots k_p}$. Consequently, the matrix $A$ can be written as a Hadamard product of two matrices:

$$A = \begin{pmatrix} a_0^1 & a_1^1 & a_2^1 & a_{12}^1 & \dots & a_{1\dots n}^1 \\ a_0^2 & a_1^2 & a_2^2 & a_{12}^2 & \dots & a_{1\dots n}^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_0^n & a_1^n & a_2^n & a_{12}^n & \dots & a_{1\dots n}^n \end{pmatrix}$$

$$= \begin{pmatrix} u_0^1 & u_1^1 & u_2^1 & u_{12}^1 & \dots & u_{1\dots n}^1 \\ u_0^2 & u_1^2 & u_2^2 & u_{12}^2 & \dots & u_{1\dots n}^2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ u_0^n & u_1^n & u_2^n & u_{12}^n & \dots & u_{1\dots n}^n \end{pmatrix} * \begin{pmatrix} 1 & r_{11} & r_{21} & (r_{11} \cdot r_{21}) & \dots & (r_{11} \cdot \dots \cdot r_{n1}) \\ 1 & r_{12} & r_{22} & (r_{12} \cdot r_{22}) & \dots & (r_{12} \cdot \dots \cdot r_{n2}) \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & r_{1n} & r_{2n} & (r_{1n} \cdot r_{2n}) & \dots & (r_{1n} \cdot \dots \cdot r_{nn}) \end{pmatrix}.$$

The first matrix in the product, denoted by $U$, is actually the matrix of the coefficients of the representation. The second matrix reflects the structural dependence of the elements of the system. It is obviously the Kronecker expansion over the rows $\overline{R^\top}$ of the transposed adjacency matrix $R$. The matrix $\overline{R^\top}$, obviously consists of zeros and ones, as does the matrix $R$. In view of this, the matrix relation can be written as follows:

$$z(t+1) = \left(U * \overline{R^\top}\right) \bar{z}(t). \tag{4}$$

This expression defines the mappings realized by a system with an arbitrary structure $H$, determined by the adjacency matrix $R$. In particular, if $H = K_n$, all elements of the matrix $R$, and therefore of the matrix $\overline{R^\top}$, are equal to one, and formula (4) coincides with formula (2).

Just as expression (3) was obtained from (2), for the coefficients of the arithmetic representation that determines the matrix $A$, from relation (4) we can express $U * \overline{R^\top}$:

$$U * \overline{R^\top} = BQ_n^{-1}. \tag{5}$$

However, unlike the previous case, in this relation the matrix $B$ cannot be arbitrary; it must be a binary matrix, but such that the elements on the right-hand side of (5) are guaranteed to be equal to zero, the locations of which are determined by the position of the zero elements of the matrix $\overline{R^\top}$.

Each state of the system can be identified with a binary vector. If we write all possible states of the system in the natural order, they form a binary matrix $G$ of size $n \times 2^n$. For $n = 2, 3$ these matrices are shown below:

$$G = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \qquad G = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Each column of the matrix $B$ is one of the possible states of the system, so we can write $B = GP$, where $P$ is a column-monomial matrix of size $2^n \times 2^n$. In other words, $GP$ is the same matrix $B$, as in the expression for the coefficients of the arithmetic representation (3), but written explicitly using the monomial matrix $P$, i.e., the set of matrices $B$ and all monomial matrices $P$ are in one-to-one correspondence. This allows us to further operate not with all possible images of binary vectors in the mapping, but with all column-monomial matrices, which is much more convenient from a formal point of view. Substituting this equality into formula (5) for the matrix of coefficients of the arithmetic representation in the case of an arbitrary structure, we obtain

$$U * \overline{R^\top} = GPQ_n^{-1}. \tag{6}$$

Further, we will call this equality the *relation for the coefficients*. If the matrix $P$ satisfies this condition for some matrix $U$, then it completely describes the binary mapping realized by the system with the given structure $H$. Thus, there is a one-to-one correspondence between mappings and monomial matrices satisfying the relation for the coefficients (6) for some matrix $U$. Therefore, each mapping can be identified with the corresponding monomial matrix $P$.

Let us denote by $\mathfrak{P}$ the set of all monomial matrices of size $2^n \times 2^n$. Obviously, $\mathfrak{P}$ is a semigroup with unity, which contains, in particular, a subsemigroup isomorphic to the symmetric group $\mathbb{S}_{2^n}$. The set $\mathcal{P}(H) \subseteq \mathfrak{P}$ of transformations $P$, satisfying the relation for the coefficients (6) for some matrix $U$, determines the *class of mappings* realized by a system with the graph structure $H$. This can be explained as follows.

The mapping of all possible states of the system is completely determined by the set of specific mappings realized by each element of this system, i.e., by the Boolean functions describing their functioning. In turn, these functions are completely determined by the corresponding coefficients of their arithmetic representations. But these coefficients are uniquely determined by the matrix $B$, and, due to what has been said above, by the corresponding monomial matrix $P$. In the case of a complete graph structure, for any possible mapping of the system states, it is possible to calculate the coefficients of the arithmetic representations of the functions of the elements, under which the system will realize this mapping.

For a system with an arbitrary structure this is not so, since the functions realized by the elements of the system, and therefore the system as a whole, are limited by the existing structure of the system, i.e., the connections between the elements. Therefore, not every mapping can be realized in such a system. Since the mapping of the states of the system is completely determined by the monomial matrix $P$, this means that the functions of the elements, and therefore the coefficients of their arithmetic representations, cannot be found for every matrix so that the relation for the coefficients (or equality) (6) is satisfied. All possible mappings that can be realized in a system with a fixed structure due to all possible choices of the functions of the elements form a class of mappings for a given structure.

Let us characterize the matrices $P$ included in the set $\mathcal{P}(H)$. For this, we will need an auxiliary "index" set. For convenience, let the matrices $\overline{R^\top}$ and $U$ be specified using the usual two-index numbering, i.e., $\overline{R^\top} = (\bar{r}_{ij})$, $U = (u_{ij})$, $i = 1, \ldots, n$, $j = 1, \ldots, 2^n$. Let us introduce the set $\mathrm{Ind}\overline{R^\top} = \{(i, j)|\ i = 1, \ldots, n,\ j = 1, \ldots, 2^n,\ \bar{r}_{ij} = 0\}$. In other words, $\mathrm{Ind}\overline{R^\top}$ is formed by all possible pairs of indices that determine the positions of the zero elements of the matrix $\overline{R^\top}$. The following statement gives a description of the set $\mathcal{P}(H)$, that does not depend on $U$.

**Statement 1.** $\mathcal{P}(H) = \left\{P|\ P \in \mathfrak{P},\ g_i P q_j^{(-1)} = 0,\ (i, j) \in \mathrm{Ind}\overline{R^\top}\right\}$.

**Proof.** Let $g_i$ and $q_j^{(-1)}$ denote the $i$th row in $G$ and the $j$th column in $Q_n^{-1}$. Then the $(i, j)$-element of the matrix in the relation for the coefficients (6) on the right is obviously equal to $g_i P q_j^{(-1)}$. If the corresponding element of the matrix $\overline{R^\top}$ is equal to zero, i.e., $\bar{r}_{ij} = 0$, then the $(i, j)$-element of the matrix $U * \overline{R^\top}$ is also equal to zero. Therefore, in order for the matrix equality for the coefficients (6) to hold, it must be $g_i P q_j^{(-1)} = 0$. Otherwise, i.e., for $\bar{r}_{ij} = 1$, the $(i, j)$-element of the matrix $U * \overline{R^\top}$ is equal to the $(i, j)$-element of the matrix $U$. Since this element is one of the coefficients, then, setting it equal to $g_i P q_j^{(-1)}$, we obtain the required equality of the elements on the left and right. Therefore, a monomial matrix $P \in \mathfrak{P}$ satisfies the relation for the coefficients (6) if it satisfies the system of relations $g_i P q_j^{(-1)} = 0$, $(i, j) \in \mathrm{Ind}\overline{R^\top}$, which proves the statement.

Each structure $H$ generates a class $\mathcal{P}(H)$ of binary mappings formed under all possible functional settings of the system elements. In particular, if $H = K_n$, then $R$ is a matrix of ones. In this case, obviously, $\mathcal{P}(K_n) = \mathfrak{P}$, and hence any mapping can be realized in such a system. If the structure of the system is a null graph, i.e., $H = O_n$, then $R = \mathbf{0}$, and $\mathcal{P}(O_n)$ contains $2^n$ matrices $P$, characterized by the fact that the elements of one of the rows are equal to one, and the rest are equal to zero.

The following statement gives an explicit expression for the cardinality $|\mathcal{P}(H)|$ of the class of mappings $\mathcal{P}(H)$, realized by a system with structure $H$.

**Statement 2.** *Let the structure of the system be defined by a directed graph $H = (V, E)$, $|V| = n$, $\rho_i$ is the indegree of a vertex $i$ in the graph $H$, $i = 1, \ldots, n$. Then*

$$|\mathcal{P}(H)| = 2^{\sum\limits_{i=1}^{n} 2^{\rho}_{i}}. \tag{7}$$

**Proof.** The number of different binary mappings realized by the $i$th element with $\rho_i$ inputs for all possible functional settings is $2^{2^{\rho_i}}$. Therefore $|\mathcal{P}(H)| = 2^{2^{\rho_1}} \cdot 2^{2^{\rho_2}} \cdot \ \ldots \ \cdot 2^{2^{\rho_n}} = 2^{\sum\limits_{i=1}^{n} 2^{\rho_i}}$.

**Corollary 1.** *1) $|\mathcal{P}(K_n)| = 2^{n2^n}$; 2) $|\mathcal{P}(O_n)| = 2^n$.*

The next section describes the classes of binary mappings that result from set-theoretic operations on systems.

## 5. OPERATIONS ON SYSTEMS AND CLASSES OF REALISABLE MAPPINGS

Let there be two systems with structures $H_1$ and $H_2$. The question arises: are there classes of mappings that, given suitable state functions of the elements, can be realized in both the first and second systems? If so, how can they be characterized? Let us examine this question.

Let us recall once again that, as described above, each mapping of possible states of a system with a given structure and fixed functions of the elements is uniquely described by a column-monomial matrix $P$. For all possible functions of the elements, a class of mappings is obtained that are realized by a system with a given structure. Therefore, the description of classes of mappings is reduced to the description of all possible and admissible monomial matrices.

If some mapping defined by the matrix $P$, is realized by both the first and the second systems, then $P \in \mathcal{P}(H_1)$ and $P \in \mathcal{P}(H_2)$, and therefore $P \in \mathcal{P}(H_1) \cap \mathcal{P}(H_2)$. The converse is also true: any matrix $P$, lying in the intersection of the sets $\mathcal{P}(H_1)$ and $\mathcal{P}(H_2)$, determines a mapping realized by each system.

**Statement 3.** *$\mathcal{P}(H_1 \cap H_2) = \mathcal{P}(H_1) \cap \mathcal{P}(H_2)$ for any $H_1$ and $H_2$.*

**Proof.** Let $R_1 = \left(r_{ij}^{(1)}\right)$ and $R_2 = \left(r_{ij}^{(2)}\right)$ be the adjacency matrices of graphs $H_1$ and $H_2$. Then $\mathcal{P}(H_1)$ is determined by the system of relations $g_i P q_j^{(-1)} = 0$, $(i, j) \in \mathrm{Ind}\overline{R_1^\top}$, and $\mathcal{P}(H_2)$ is determined by the system of relations $g_i P q_j^{(-1)} = 0$, $(i, j) \in \mathrm{Ind}\overline{R_2^\top}$. If $P \in \mathcal{P}(H_1) \cap \mathcal{P}(H_2)$, then this means that $P$ must satisfy each of these systems or, in other words, must satisfy the system of relations $g_i P q_j^{(-1)} = 0$, $(i, j) \in \mathrm{Ind}\overline{R_1^\top} \cup \mathrm{Ind}\overline{R_2^\top}$.

Let us introduce the matrix $\overline{R^\top}$, which is defined as follows: $(i, j)$-element of this matrix is zero if the corresponding element of at least one of the matrices $\overline{R_1^\top}$ or $\overline{R_2^\top}$, is zero, and one when both elements are one. Then, obviously, $\mathrm{Ind}\overline{R^\top} = \mathrm{Ind}\overline{R_1^\top} \cup \mathrm{Ind}\overline{R_2^\top}$. It follows directly from the definition of the matrix $\overline{R^\top}$ that $\overline{R^\top} = \overline{R_1^\top} * \overline{R_2^\top}$. Let us now show that $\overline{R^\top}$ is a Kronecker extension along the rows of the matrix $R^\top = (R_1 * R_2)^\top = R_1^\top * R_2^\top$, i.e., $\overline{R^\top} = \overline{R_1^\top * R_2^\top}$. To do this, it suffices to prove the relation

$$\overline{R_1^\top * R_2^\top} = \overline{R_1^\top} * \overline{R_2^\top}. \tag{8}$$

First of all, note that the first columns of these matrices are equal, since they contain only ones. Let us show that the remaining elements are also equal. An arbitrary element of the matrix on the left has the form $\left(r_{k_1 i}^{(1)} r_{k_1 i}^{(2)}\right) \cdot \ldots \cdot \left(r_{k_p i}^{(1)} r_{k_p i}^{(2)}\right)$, $p = 1, \ldots, n$, and the corresponding element of the right matrix is $\left(r_{k_1 i}^{(1)} \cdot \ldots \cdot r_{k_p i}^{(1)}\right) \cdot \left(r_{k_1}^{(2)} \cdot \ldots \cdot r_{k_p i}^{(2)}\right)$. Both expressions differ only in the order of the factors, and therefore are equal. Therefore, equality (8) is true. But $R = R_1 * R_2$ is the adjacency matrix of the graph $H = H_1 \cap H_2$, from which the validity of the statement follows.

**Corollary 2.** *$\mathcal{P}(H_1 \cap \ \dots \ \cap H_k) = \mathcal{P}(H_1) \cap \ \dots \ \cap \mathcal{P}(H_k)$ for any $H_1, \ \dots \ , H_k$.*

This corollary is proved by induction. Statement 3 establishes a precise connection between classes of binary mappings and systems at their intersection. A similar relation for the union of systems is generally incorrect, but the following inclusion holds.

**Statement 4.** *$\mathcal{P}(H_1 \cup H_2) \supseteq \mathcal{P}(H_1) \cup \mathcal{P}(H_2)$ for any $H_1$ and $H_2$.*

**Proof.** Using Statement 3, we have

$$\mathcal{P}(H_1 \cup H_2) \cap \mathcal{P}(H_1) = \mathcal{P}\left((H_1 \cup H_2) \cap H_1)\right) = \mathcal{P}(H_1).$$

It follows that $\mathcal{P}(H_1) \subseteq \mathcal{P}(H_1 \cup H_2)$. Similarly, we obtain $\mathcal{P}(H_2) \subseteq \mathcal{P}(H_1 \cup H_2)$. The validity of the statement follows from these two inclusions.

The substantive meaning of Statement 4 is that the set of mapping classes realized by a system with structure $H = H_1 \cup H_2$, is greater than the simple union of mapping classes of the systems being combined. This fact, in a certain sense, reflects the "superadditive effect" inherent in systems, corresponding to the thesis "the whole is greater than the sum of its parts." The mapping classes realized in a system with a graph structure $H$ and not realized in any of the systems being combined demonstrate the effect of emergence, understood in the narrow sense. This statement can be easily generalized to any number of systems.

**Corollary 3.** *$\mathcal{P}(H_1 \cup \ \dots \ \cup H_k) \supseteq \mathcal{P}(H_1) \cup \ \dots \ \cup \mathcal{P}(H_k)$ for any $H_1, \ \dots \ , H_k$.*

From Statement 4 follows the property of monotonicity with respect to inclusion of systems.

**Corollary 4.** *It follows from $H_1 \subseteq H_2$ that $\mathcal{P}(H_1) \subseteq \mathcal{P}(H_2)$.*

Let us characterize in more detail the left and right parts of Statement 4.

**Statement 5.** *The set $\mathcal{P}(H_1 \cup H_2)$ consists of those and only those matrices $P \in \mathfrak{P}$, that satisfy the system of relations $g_i P q_j^{(-1)} = 0$, $(i, j) \in \mathrm{Ind}\overline{(R_1 + R_2 - R_1 * R_2)^\top}$.*

**Proof.** Let $H = H_1 \cup H_2$. It is obvious that the adjacency matrix $R$ of the structure $H$ is connected with the adjacency matrices $R_1$ and $R_2$ of the structures $H_1$ and $H_2$ by the equality $R = R_1 + R_2 - R_1 * R_2$. In view of this, the validity of the statement being proved follows from Statement 1.

**Statement 6.** *The set $\mathcal{P}(H_1) \cup \mathcal{P}(H_2)$ consists of those and only those matrices $P \in \mathfrak{P}$, that satisfy the system of relations*

$$g_i P q_j^{(-1)} g_k P q_l^{(-1)} = 0, \quad (i, j) \in \mathrm{Ind}\overline{R_1^\top}, \quad (k, l) \in \mathrm{Ind}\overline{R_2^\top}. \tag{9}$$

**Proof.** If $P \in \mathcal{P}(H_1)$ or $P \in \mathcal{P}(H_2)$, then $P$ satisfies the system of relations (9). Conversely, if $P$ does not belong to, for example, $\mathcal{P}(H_1)$, but satisfies (9), then for some $(i, j) \in \mathrm{Ind}\overline{R_1^\top}$ we have $g_i P q_j^{(-1)} \neq 0$. If we consider the equations of the system of relations (9), in which the indices $i$ and $j$ coincide with the chosen ones, i.e., $g_i P q_j^{(-1)} g_k P q_l^{(-1)} = 0$, $(k, l) \in \mathrm{Ind}\overline{R_2^\top}$, then we obtain the relations $g_k P q_l^{(-1)} = 0$, $(k, l) \in \mathrm{Ind}\overline{R_2^\top}$. Consequently, $P \in \mathcal{P}(H_2)$. Similarly, the matrix $P$, which does not belong to $\mathcal{P}(H_2)$ and satisfies (9), belongs to $\mathcal{P}(H_1)$, which proves the statement.

## 6. EMERGENCE COEFFICIENT

As shown above, when combining systems, a "superadditive" property appears, which is expressed in the fact that the new system is capable of realizing mappings that are not realizable by any of the original systems. In order to quantitatively characterize this effect, it is necessary to introduce a suitable scalar indicator that can be interpreted as the degree of irreducibility of the properties of the system to the properties of its elements.

The simplest and most straightforward approach is to use the ratio $\frac{|\mathcal{P}(H_1 \cup H_2)|}{|\mathcal{P}(H_1) \cup \mathcal{P}(H_2)|}$, which shows the multiplicity of the excess of the number of implemented mappings when combining systems. However, despite its simplicity, such an indicator is difficult to interpret. This is due to the fact that it takes very large values due to the twice exponential growth of the cardinality of the classes of these mappings, as established in Statement 2. The following considerations suggest a slightly different approach to constructing such an indicator. Let us consider simple examples.

Let the graph of the structure of a three-element system have $n = 3$ vertices and $\rho_1 = \rho_2 = 1$, $\rho_3 = 2$ be the indegrees of its vertices. Then, by virtue of Statement 2, the cardinality of the class of realizable mappings is equal to $2^{2^1+2^1+2^2} = 2^8 = 256$. On the other hand, the class of mappings realizable by a system with a structure $K_2$ with two elements will have exactly this cardinality, i.e., $2^{2\cdot 2^2} = 2^8 = 256$.

Another similar example. The graph $H$ has $n = 6$ vertices with indegrees $2, 1, 1, 2, 3, 2$. Then $|\mathcal{P}(H)| = 2^{2^2+2^1+2^1+2^2+2^3+2^2} = 2^{24}$. But the class of mappings of the same power is realized by the structure $K_3$, i.e., $|\mathcal{P}(K_3)| = 2^{3\cdot 2^3} = 2^{24} = |\mathcal{P}(H)|$. In other words, in each of these cases for some $\mu$ the equality $2^{\mu 2^\mu} = 2^{2^{\rho_1}+2^{\rho_2}+\ldots+2^{\rho_n}}$ is satisfied or

$$\mu 2^\mu = 2^{\rho_1} + 2^{\rho_2} + \ldots + 2^{\rho_n}. \tag{10}$$

In these examples, $\mu$ is the smallest number of vertices of a complete graph with loops that realizes exactly the same number of mappings as some graph with given indegrees of vertices. In the cases considered, $\mu$ was an integer and could be interpreted as the number of vertices of a complete directed graph. In the general case, the solution $\mu$ of equation (10) for arbitrary $\rho_1, \rho_2, \ldots, \rho_n$ will not be an integer. Nevertheless, speculatively reasoning, we can assume that the solution of equation (10) is a scalar characteristic of the considered class of mappings as a minimal, but “fuzzy” (not integer) number of vertices of a complete directed graph with loops that realizes the same number of different mappings. Since the function $\mu 2^\mu$ is positive and monotonically increasing on $[0, \infty)$, then for any right-hand side, equation (10) has a unique root.

The quantity $\mu$ can be considered as a measure $\mu = \mu\,[\mathcal{P}(H)]$ of a set $\mathcal{P}(H)$ on the Boolean $2^{\mathfrak{P}}$, where $\mathfrak{P}$, as defined above, is the set of all monomial matrices of the corresponding size. This measure is obviously not additive, but is *monotone* with respect to inclusion of classes, since $\mu\,[\mathcal{P}(H_1)] \leqslant \mu\,[\mathcal{P}(H_2)]$ for $\mathcal{P}(H_1) \subseteq \mathcal{P}(H_2)$.

By the *emergence coefficient* $\kappa\,(H_1, H_2)$ when uniting systems $H_1$ and $H_2$ we will call the quantity $\kappa(H_1, H_2) = \frac{\mu[\mathcal{P}(H_1 \cup H_2)]}{\mu[\mathcal{P}(H_1) \cup \mathcal{P}(H_2)]}$. From this definition it follows that $\mu\,[\mathcal{P}(H_1 \cup H_2)] = \mu_1$ and $\mu\,[\mathcal{P}(H_1) \cup \mathcal{P}(H_2)] = \mu_2$ are the roots of the corresponding equations

$$\mu_1 2^{\mu_1} = \log_2 |\mathcal{P}(H_1 \cup H_2)|\,, \tag{11}$$

$$\mu_2 2^{\mu_2} = \log_2 |\mathcal{P}(H_1) \cup \mathcal{P}(H_2)| = \log_2 (|\mathcal{P}(H_1)| + |\mathcal{P}(H_2)| - |\mathcal{P}(H_1 \cap H_2)|)\,. \tag{12}$$

The introduced concept of the emergence coefficient can easily be generalized to an arbitrary number of uniting systems. The following theorem gives two-sided estimates of the introduced coefficient.

**Theorem 1.** *The emergence coefficient $\kappa\,(H_1, H_2)$ for the union of any two $n$-element systems $H_1$ and $H_2$ satisfies the inequalities*

$$1 \leqslant \kappa(H_1, H_2) < 1 + \sqrt{1 + \frac{1}{e \ln 2}} \approx 2.2372.$$

The obtained upper bound is quite accurate, which is especially noticeable as $n$ increases. The table below shows the values of the maximum emergence coefficient for $n = 2, \dots, 20$ (accurate to four decimal places). Comparison of the given data with the found upper bound allows us to make an assumption about its asymptotic accuracy.

**Table 1.** Values of the maximum emergence coefficient

| $n$ | Maximum emergence coefficient | $n$ | Maximum emergence coefficient |
|---|---|---|---|
| | | 11 | 2.0822 |
| 2 | 1.2553 | 12 | 2.1245 |
| 3 | 1.3729 | 13 | 2.1567 |
| 4 | 1.4979 | 14 | 2.1807 |
| 5 | 1.5981 | 15 | 2.1980 |
| 6 | 1.6946 | 16 | 2.2102 |
| 7 | 1.7869 | 17 | 2.2184 |
| 8 | 1.8803 | 18 | 2.2236 |
| 9 | 1.9610 | 19 | 2.2265 |
| 10 | 2.0281 | 20 | 2.2278 |

## 7. CONCLUSION

The paper proposes an apparatus for studying system effects arising in arbitrary binary mapping systems. Despite the binary nature of the implemented mappings, the standard algebraic technique is used for the study, which simplifies the description and understanding of emergence effects in the narrow sense. The analysis allows us to conclude that the classes of mappings observed in such systems and their transformations during set-theoretical operations on the structures of these systems are similar to the "expansion effect" when combining classical algebraic structures. The results obtained, in particular, Statements 3, 4 and their consequences, as well as estimates for the emergence coefficient given by Theorem 1, allow us to establish in which cases and what quantitatively the emergent effect can be achieved when aggregating binary systems. This can be used, for example, to optimize the collective behavior of individual groups of functioning objects or subjects.

The model of binary mappings considered in the work is a fairly narrow class of mappings. However, due to its relative simplicity, it allows us to identify some patterns inherent in system interactions and even characterize them quantitatively. This, in fact, was the motivation for this work. Generalization of these studies to a more general class of system interactions (even in the autonomous case) will require the use of a more complex mathematical apparatus associated with morphisms on differential manifolds and the construction of suitable measures on such objects. This is the topic of a separate complex study. But such a theory can provide a more natural description of emergence effects in physical environments and in technical systems.

*APPENDIX*

**Proof of Theorem 1.** The left inequality follows immediately from Statement 4 and the monotonicity of the function $x2^x$. Let us prove the right inequality.

Let $H_1 = (V, E_1)$ and $H_2 = (V, E_2)$ be arbitrary directed graphs and $|V| = n$. Consider the sets of arcs $E_C = E_1 \cap E_2$, $E_A = E_1 \backslash E_C$ and $E_B = E_2 \backslash E_C$. These sets are obviously pairwise disjoint. Let us construct three graphs $H_A = (V, E_A)$, $H_B = (V, E_B)$ and $H_C = (V, E_C)$; let $\rho_i^A, \rho_i^B, \rho_i^C$, $i = 1, \dots, n$, denote the indegrees of their vertices, respectively. Then $H_1 = H_A \cup H_C =$

$(V, E_A \cup E_C)$, $H_2 = H_B \cup H_C = (V, E_B \cup E_C)$ and by virtue of Statement 2, equations (11) and (12) can be written as follows:

$$\mu_1 2^{\mu_1} = \sum_{i=1}^{n} 2^{\rho_i^A+\rho_i^B+\rho_i^C}, \qquad \mu_2 2^{\mu_2} = \log_2 \left( 2^{\sum\limits_{i=1}^{n} 2^{\rho_i^A+\rho_i^C}} + 2^{\sum\limits_{i=1}^{n} 2^{\rho_i^B+\rho_i^C}} - 2^{\sum\limits_{i=1}^{n} 2^{\rho_i^C}} \right).$$

Let $\rho_i^A + \rho_i^B + \rho_i^C = m_i$ for some fixed $m_i \leqslant n$, $i = 1, \ldots, n$. In this case, the right-hand side in the first equation is a constant value, and, therefore, due to the monotonicity of the function $x2^x$ the emergence coefficient $\kappa$ will reach its maximum value at the smallest value of the right-hand side of the second equation. To find such a value, it is necessary to solve the problem

$$\varphi = 2^{\sum\limits_{i=1}^{n} 2^{m_i-\rho_i^A}} + 2^{\sum\limits_{i=1}^{n} 2^{m_i-\rho_i^B}} - 2^{\sum\limits_{i=1}^{n} 2^{m_i-\rho_i^A-\rho_i^B}} \xrightarrow{\rho_i^A,\, \rho_i^B} \min.$$

Let us find the stationary points of the function $\varphi$. Having calculated the partial derivatives with respect to $\rho_j^A$, $\rho_j^B$, $j = 1, \ldots, n$, and equating them to zero, after equivalent transformations we obtain:

$$2^{\rho_j^B} \cdot 2^{\sum\limits_{i=1}^{n} 2^{m_i-\rho_i^A}} = 2^{\sum\limits_{i=1}^{n} 2^{m_i-\rho_i^A-\rho_i^B}}, \quad 2^{\rho_j^A} \cdot 2^{\sum\limits_{i=1}^{n} 2^{m_i-\rho_i^B}} = 2^{\sum\limits_{i=1}^{n} 2^{m_i-\rho_i^A-\rho_i^B}}, \quad j = 1, \ldots, n. \tag{A.1}$$

Since the right-hand sides are equal, then $2^{-\rho_j^A} \cdot 2^{\sum\limits_{i=1}^{n} 2^{m_i-\rho_i^A}} = 2^{-\rho_j^B} \cdot 2^{\sum\limits_{i=1}^{n} 2^{m_i-\rho_i^B}}$, $j = 1, \ldots, n$. Let us denote $X = \sum\limits_{i=1}^{n} 2^{m_i-\rho_i^A}$, $Y = \sum\limits_{i=1}^{n} 2^{m_i-\rho_i^B}$ and rewrite the expression as:

$$2^{m_j-\rho_j^A} \cdot 2^X = 2^{m_j-\rho_j^B} \cdot 2^Y. \tag{A.2}$$

Let us sum the last equality over $j$, and obtain: $X2^X = Y2^Y$. The function $x2^x$ strictly monotonically increases at $x \geqslant 0$, therefore the last equation has a unique solution $X = Y$, and therefore, from (14) we obtain that $\rho_j^A = \rho_j^B = \rho_j$, $j = 1, \ldots, n$. Let us substitute these conditions into any of the equations (13), and after taking the logarithm and rearranging the terms we obtain: $\rho_j + \sum\limits_{i=1}^{n} \left(2^{m_i-\rho_i} - 2^{m_i-2\rho_i}\right) = 0$. At $\rho_i \geqslant 0$ all terms on the left-hand side are non-negative, therefore the last equality is possible only when all these terms are equal to zero, whence it follows that all $\rho_i = 0$. In the admissible region, this is the only stationary point at which an extremum can be reached. This point lies on the boundary of the region, and the value of the function at it is equal to $\varphi = 2^{\sum\limits_{i=1}^{n} 2^{m_i}}$. However, using the standard technique of studying an extremum using second derivatives, it can be shown that the quadratic form describing the second differential of the function $\varphi$ at the found point is sign-indefinite, i.e., there is no extremum at this point. This means that the function $\varphi$ takes an extremal value on the boundary of the region $\rho_i^A, \rho_i^B \geqslant 0$, $\rho_i^A + \rho_i^B \leqslant m_i$, i.e., under the condition $\rho_i^A, \rho_i^B \geqslant 0$, $\rho_i^A + \rho_i^B = m_i$, $i = 1, \ldots, n$.

To find a stationary point on the boundary, we substitute $\rho_i^B = m_i - \rho_i^A$ into the expression for the function $f$: $f = 2^{\sum\limits_{i=1}^{n} 2^{m_i-\rho_i^A}} + 2^{\sum\limits_{i=1}^{n} 2^{\rho_i^A}} - 2^n$. Having calculated the derivatives with respect to $\rho_j^A$ and equated them to zero, after transformations we obtain a system of equations for finding stationary points:

$$2^{\sum\limits_{i=1}^{n} 2^{m_i-\rho_i^A}} \cdot 2^{m_j-\rho_j^A} = 2^{\sum\limits_{i=1}^{n} 2^{\rho_i^A}} \cdot 2^{\rho_j^A}, \qquad j = 1, \ldots, n. \tag{A.3}$$

We sum up all the equations with respect to $j$ and denote $X = \sum\limits_{i=1}^{n} 2^{m_i-\rho_i^A}$ and $Y = \sum\limits_{i=1}^{n} 2^{\rho_i^A}$, as a result we obtain $X2^X = Y2^Y$. This equality is possible only for $X = Y$. Therefore, from (15) we obtain $2^{m_j-\rho_j^A} = 2^{\rho_j^A}$, whence $\rho_j^A = \frac{m_j}{2} = \rho_j^B$, $j = 1,\dots,n$. Routine research using second derivatives shows that at this point the minimum of the function $\varphi_{\min} = 2^{1+\sum_{i=1}^{n} 2^{\frac{m_i}{2}}} - 2^n$ is indeed reached. Thus, to find the greatest value of the emergence coefficient $\kappa$ under the condition $\rho_i^A+\rho_i^B+\rho_i^C = m_i$ we obtain the equations:

$$\mu_1 2^{\mu_1} = \sum_{i=1}^{n} 2^{m_i}, \tag{A.4}$$

$$\mu_2 2^{\mu_2} = \log_2\left(2\cdot 2^{\sum_{i=1}^{n} 2^{\frac{m_i}{2}}} - 2^n\right). \tag{A.5}$$

Let us estimate the right-hand side of (17) from below. Since $2^{\sum_{i=1}^{n} 2^{\frac{m_i}{2}}} - 2^n \geqslant 0$ for any $m_i$, $i = 1,\dots,n$, then

$$\log_2\left(2\cdot 2^{\sum_{i=1}^{n} 2^{\frac{m_i}{2}}} - 2^n\right) \geqslant \log_2 2^{\sum_{i=1}^{n} 2^{\frac{m_i}{2}}} = \sum_{i=1}^{n} 2^{\frac{m_i}{2}} > \left(\sum_{i=1}^{n} 2^{m_i}\right)^{\frac{1}{2}}.$$

Along with equation (17), we consider equation $\bar\mu_2 2^{\bar\mu_2} = \left(\sum_{i=1}^{n} 2^{m_i}\right)^{\frac{1}{2}}$. Due to the monotonicity of the function $x2^x$ we conclude that $\bar\mu_2 < \mu_2$. Consequently, $\kappa = \frac{\mu_1}{\mu_2} < \frac{\mu_1}{\bar\mu_2} = \bar\kappa$, so $\bar\kappa$ can be considered as an upper bound for $\kappa$.

Now we divide equation (16) by $(\bar\mu_2 2^{\bar\mu_2})^2$: $\frac{\mu_1 2^{\mu_1}}{\bar\mu_2^2 2^{2\bar\mu_2}} = \frac{\bar\kappa}{\bar\mu_2} 2^{\bar\mu_2(\bar\kappa-2)} = 1$. We consider the last equation as an implicitly defined function $\bar\kappa$, depending on $\bar\mu_2$, i.e., $F(\bar\mu_2,\bar\kappa) = \frac{\bar\kappa}{\bar\mu_2} 2^{\bar\mu_2(\bar\kappa-2)} - 1 = 0$. We find the extremum of the implicit function $\bar\kappa$ of $\bar\mu_2$. In order for $\bar\kappa'_{\bar\mu_2} = 0$, the equality $F'_{\bar\mu_2} = -\frac{\bar\kappa}{\bar\mu_2^2} 2^{\bar\mu_2(\bar\kappa-2)} + \frac{\bar\kappa}{\bar\mu_2} 2^{\bar\mu_2(\bar\kappa-2)}(\bar\kappa-2)\ln 2 = 0$ must be satisfied. Solving the equations $F'_{\bar\mu_2} = 0$ and $F = 0$, together, we find two solutions to these equations: $\bar\kappa_{1,2} = 1 \pm \sqrt{1+\frac{1}{e\ln 2}}$. Since the emergence coefficient is obviously a positive value, we finally obtain $\kappa < \bar\kappa = 1+\sqrt{1+\frac{1}{e\ln 2}} \approx 2.2372$. Using standard methods, we can show that the found value is the only maximum of the function under study, which proves the upper bound for $\kappa$.

Theorem 1 is proven.

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